Theoretical Computer Science 71 (1990) 425–429
North-Holland

NOTE

NON-RECURSIVENESS OF THE OPERATIONS ON REAL NUMBERS

Kostas SKANDALIS

Department of Mathematics, University of Crete, Greece

Communicated by M. Nivat
Received April 1988

Abstract. For all known representations of the real numbers the basic operations on real numbers are *not* recursive. In this paper we prove that this is true for *every* effective representation of the real numbers.

Introduction

We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers. By a *representation* of $\mathbb{R}$ we mean any one-to-one mapping from the set $\mathbb{R}$ to the Baire space $\mathbb{N}^{\mathbb{N}}$. All known representations of $\mathbb{R}$ can be described as mappings $V:\mathbb{R} \to^{1\text{-}1} \mathbb{N}^{\mathbb{N}}$, i.e. to each real number corresponds a sequence of natural numbers which is its "name". Sometimes the representations are defined as "numberings" $\Lambda:\mathbb{N}^{\mathbb{N}} \to^{\text{onto}} \mathbb{R}$ (cf. [4, 7, 8]). However, in these cases there exist methods of finding names for the real numbers, i.e. "effective" functions $V:\mathbb{R} \to^{1\text{-}1} \mathbb{N}^{\mathbb{N}}$ such that $\Lambda(V(x)) = x$ for all $x \in \mathbb{R}$.

For any representation V of $\mathbb{R}$ one can consider the basic operations of real numbers as operations on names of the real numbers. For instance, the addition of real numbers corresponds to a (partial) function $\oplus_V : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$\oplus_V(V(x), V(y)) = V(x+y) \quad \text{for all } x, y \in \mathbb{R}.$$

For all known classical representations V the function $\oplus_V$ is not recursive in the sense of second order recursion (cf. [3]). The function $\oplus_V$ has not finitistic character, there is no method to compute the nth coordinate of the name of $x+y$ using initial segments of the names of x and y.

One can ask if there exists a representation of the real numbers for which the operations on $\mathbb{R}$ become recursive. The answer is negative for all "effective" representations. Effectiveness means here that there exists a method which allows us to find the names of real numbers. More precisely, we call a function $V:\mathbb{R} \to \mathbb{N}^{\mathbb{N}}$ effective if there exists a method which, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, finds in finitely many steps the nth coordinate of the "name" of x, i.e. the value $V(x)(n)$. This means that the function $V^*(x, n) = V(x)(n)$ is computable by some program.[1]

A rough idea of the notion of programmability in the ordered field of real numbers is described in Section 1. More details about programmability in abstract structures can be found for instance in [1, 6]. In Section 2 we prove that there is no representation V of $\mathbf{R}$ with programmable V^* which makes the addition of real numbers a recursive operation. In Section 3 we remark that similar results hold for the other operations on the real numbers.

1. Programmability

We describe the notion of *program* for the structure $\mathbf{RO} = \langle \mathbf{R}, +, -, \cdot, :, <, 0, 1\rangle$. The simplest programs are the expressions of the form $v := t$ where v is a variable and t is a term (of the first order language for the above structure). More complicated programs are obtained by the following three programming schemata. Let M, N be programs and φ an open formula (first order formula without quantifiers). Then the expressions:

begin M; N **end**
if φ **then** M **else** N
while φ **do** M

are also programs.

One can define in a natural way the *interpretation* of a program in the structure $\mathbf{RO}$ (see [1, 6]). A (partial) real function is called *programmable* if it is an interpretation of some program.

One can prove (cf. [6]) that for any programmable function in $\mathbf{RO}$ there exist sequences $(t_j)_{j\in\mathbf{N}}$ of terms and $(\varphi_j)_{j\in\mathbf{N}}$ of open formulae such that f can be defined by an infinite sequence of cases (the so-called Friedman's schema, [2]), i.e.

$$f(x) = \begin{cases} \tilde{t}_0(x) & \text{if } \varphi_0(x) \text{ holds in } \mathbf{RO} \\ \tilde{t}_1(x) & \text{if } \varphi_1(x) \text{ holds in } \mathbf{RO} \\ \vdots & \vdots \\ \tilde{t}_j(x) & \text{if } \varphi_j(x) \text{ holds in } \mathbf{RO} \\ \vdots & \vdots \end{cases}$$

where $\tilde{t}_j$ is the rational function which is the interpretation of the term t_j.

2. Non-recursiveness of the operations on R

Let $V : \mathbf{R} \to^{1\text{-}1} \mathbf{N}^{\mathbf{N}}$ be a representation of the real numbers and consider the function $V^*(x, n) = V(x)(n)$. We assume that V^* is effective, i.e. it is an interpretation of

[1] Readers not familiar with abstract programmability, or disagreeing with such a point of view to effectiveness on the real line, can think of this paper as purely mathematical. They can avoid the concept of programmability by taking the conclusion of Lemma 2.1 as an initial hypothesis and continue directly to Lemma 2.2.

some program. To prove the non-recursiveness of the addition of the names of reals we first need some lemmas.

Lemma 2.1. *For any finite sequence $m_0m_1 \dots m_k$ of natural numbers the set $A(m_0m_1 \dots m_k) = \{x \in \mathbf{R}: m_0m_1 \dots m_k$ is the k-prefix[2] of $V(x)\}$ is an F_σ-set.*

Proof. If the function V^* is programmable, then the functions V_n $(n \in \mathbf{N})$ with $V_n(x) = V^*(x, n)$ are also programmable. Fix $n \in \mathbf{N}$ and let $(t_j)_{j \in \mathbf{N}}$, $(\varphi_j)_{j \in \mathbf{N}}$ be sequences of terms and open formulae, respectively, which define V_n by Friedman's schema. For each j, the interpretation $\tilde{t}_j$ of the term t_j is a rational function and hence for any $m \in \mathbf{N}$ the set

$$\{x \in \mathbf{R}: \tilde{t}_j(x) = m\}$$

is closed. One can also easily prove that for any open formula φ, the set

$$\{x \in \mathbf{R}: \varphi(x) \text{ holds in } \mathbf{RO}\}$$

is an F_σ-set. It follows that the set

$$\begin{aligned} B(n, m) &= \{x \in \mathbf{R}: V_n(x) = m\} \\ &= \bigcup_{j \in \mathbf{N}} (\{x \in \mathbf{R}: \tilde{t}_j(x) = m\} \cap \{x \in \mathbf{R}: \varphi_j(x) \text{ holds}\}) \end{aligned}$$

is an F_σ-set. We complete the proof by noting that

$$A(m_0m_1 \dots m_k) = B(0, m_0) \cap B(1, m_1) \cap \cdots \cap B(k, m_k). \qquad \square$$

Lemma 2.2. *There exist $a \in \mathbf{R}$, $(m_k)_{k \in \mathbf{N}} \in \mathbf{N}^{\mathbf{N}}$ and a sequence $(I_k)_{k \in \mathbf{N}}$ of open finite intervals such that*

(i) $V(a) = (m_k)_{k \in \mathbf{N}}$

and for any $k \in \mathbf{N}$

(ii) $a \in I_k$,
(iii) $\bar{I}_l \subseteq A(m_0m_1 \dots m_k)$,
(iv) $\bar{I}_{k+1} \subseteq I_k$.

Proof. We first define by induction the sequences $(m_k)_{k \in \mathbf{N}}$, $(I_k)_{k \in \mathbf{N}}$. Let us note that $\mathbf{R} = \bigcup_{m \in \mathbf{N}} A(m)$, i.e. $\mathbf{R}$ is covered by F_σ-sets (Lemma 2.1). Baire's Category Theorem shows that there exist an $m_0 \in \mathbf{N}$ and an open interval J such that $\emptyset \neq J \subseteq A(m_0)$. We can choose a smaller interval I_0 satisfying: $\emptyset \neq \bar{I}_0 \subseteq J \subseteq A(m_0)$.

Suppose now that $m_0, m_1, \dots, m_n$ and $I_0, I_1, \dots, I_n$ have been defined so that (iii) holds for $k = 0, 1, \dots, n$ and (iv) holds for $k = 0, 1, \dots, n-1$. Since we have

$$\emptyset \neq \bar{I}_n \subseteq A(m_0m_1 \dots m_n) = \bigcup_{m \in \mathbf{N}} A(m_0m_1 \dots m_nm)$$

[2] This of course means that $V(x)(j) = m_j$ for $j = 0, 1, \dots, k$.

the open interval I_n is covered by F_σ-sets. Applying again Baire's Category Theorem we conclude that there exist $m_{n+1} \in \mathbf{N}$ and a non-empty open interval J, $J \subseteq I_n \cap A(m_0 m_1 \dots m_n m_{n+1})$. As above we may choose an interval I_{n+1}, $\bar{I}_{n+1} \subseteq J$ and hence satisfying

$$\bar{I}_{n+1} \subseteq A(m_0 m_1 \dots m_n m_{n+1}) \quad \text{and} \quad \bar{I}_{n+1} \subseteq I_n.$$

This concludes the induction for the definition of the sequences $(m_k)_{k\in\mathbf{N}}$, $(I_k)_{k\in\mathbf{N}}$.

It is clear that if $x \in \bigcap_{k\in\mathbf{N}} A(m_0 m_1 \dots m_k)$, then for each $k \in \mathbf{N}$, the name $V(x)$ has k-prefix the sequence $m_0 m_1 \dots m_k$. Since the intersection $\bigcap_{k\in\mathbf{N}} \bar{I}_k$ of the compact intervals $\bar{I}_k$ is nonempty, it follows that the intersection $\bigcap_{k\in\mathbf{N}} A(m_0 m_1 \dots m_k)$ has exactly one point. Let us denote this point by a. We clearly have $V(a) = (m_k)_{k\in\mathbf{N}}$.

It remains to prove that $a \in I_k$ for any $k \in \mathbf{N}$. By the already proved property (iv), we have

$$\{a\} = \bigcap_{k\in\mathbf{N}} A(m_0 m_1 \dots m_k) = \bigcap_{k\in\mathbf{N}} \bar{I}_k = \bigcap_{k\in\mathbf{N}} I_k,$$

which proves (ii). □

We are now ready to prove the theorem on the non-recursiveness of the addition. Since it is well known that recursive functions on the Baire space are continuous (cf. [3, 5]) it is enough to prove that $\oplus_V$ is not continuous. We shall prove that there is no continuous extension of $\oplus_V$.

Theorem 2.3. *Let $V : \mathbf{R} \to^{1\text{-}1} \mathbf{N}^{\mathbf{N}}$ be a representation of the real numbers. Let us assume that the function V^* defined by $V^*(x, n) = V(x)(n)$ is programmable. Then there is no partial continuous function $\Gamma : \mathbf{N}^{\mathbf{N}} \times \mathbf{N}^{\mathbf{N}} \to \mathbf{N}^{\mathbf{N}}$* such that for all $x, y \in \mathbf{R}$

$$\Gamma(V(x), V(y)) = V(x+y).$$

Proof. Suppose that such a Γ exists. Let $a \in \mathbf{R}$, $(m_k)_{k\in\mathbf{N}}$ and $(I_k)_{k\in\mathbf{N}}$ be as in Lemma 2.2. For any $k \in \mathbf{N}$ define

$$(*) \qquad p_k = \sup\{z \in \mathbf{R} : [a, z] \subseteq A(m_0 m_1 \dots m_k)\}.$$

Since $\bigcap_{k\in\mathbf{N}} A(m_0 m_1 \dots m_k) = \{a\}$ and $[a, p_k) \subseteq A(m_0 m_1 \dots m_k)$ there exists k such that $p_k < +\infty$. For notational simplicity we write $p_k = p$. Let $l_0 l_1 \dots l_k$ be the k-prefix of $V(p)$.

For $r = p - a$ we have $\Gamma(V(a), V(r)) = V(p)$. By continuity of Γ there exists $n \in \mathbf{N}$ such that for any ξ with $(\xi, V(r)) \in \operatorname{dom} \Gamma$:

$(**)$ if ξ has n-prefix $m_0 m_1 \dots m_n$, then $\Gamma(\xi, V(r))$ has k-prefix $l_0 l_1 \dots l_k$.

By the definition $(*)$ of p and the fact $a \in I_n$ we can choose $\varepsilon > 0$ such that

$$[p-\varepsilon, p) \subseteq A(m_0 m_1 \dots m_k) \quad \text{and} \quad [a-\varepsilon, a+\varepsilon] \subseteq I_n \subseteq A(m_0 m_1 \dots m_n).$$

For any $x \in [a-\varepsilon, a+\varepsilon]$ we have that $m_0 m_1 \dots m_n$ is the n-prefix of $V(x)$. It follows from $(**)$ that for any such x, $V(x+r)$ has k-prefix $l_0 l_1 \dots l_k$. This means that the interval $[p-\varepsilon, p+\varepsilon]=[a-\varepsilon+r, a+\varepsilon+r]$ is included in $A(l_0 l_1 \ldots l_k)$. But $p-\varepsilon$ belongs to $A(m_0 m_1 \ldots m_k)$, hence the sequences $m_0 m_1 \ldots m_k$ and $l_0 l_1 \ldots l_k$ are equal. Therefore, the interval $[a, p+\varepsilon]$ is included in $A(m_0 m_1 \ldots m_k)$, which contradicts the definition (*) of the point p. □

3. Final remarks

As a corollary of the above theorem we have a similar result for the operation of subtraction. Suppose that there exists a partial continuous function Δ such that for all $x, y \in \mathbb{R}$

$$\Delta(V(x), V(y)) = V(x-y).$$

If we put $\Gamma(\alpha, \beta) = \Delta(\alpha, \Delta(V(0), \beta))$, we have a continuous function Γ with $\Gamma(V(x), V(y)) = \Delta(V(x), \Delta(V(0), V(y))) = V(x+y)$, which contradicts Theorem 2.3.

We can also prove similar results for the operations of multiplication and division of real numbers. The proofs follow the same line as that for the operation of addition. Briefly, one can work as follows. A slight modification of Lemma 2.2 gives $a \neq 0$ (it suffices to choose I_0 so that $0 \notin I_0$). On observing that p_k converges to a, we see that if k is sufficiently large then $a \cdot p_k > 0$. We fix such k and put $p = p_k$. We choose now r such that $a \cdot r = p$ (for the multiplication) $a : r = p$ (for the division) and continue essentially as before.

References

[1] L. Banachowski, A. Kreczmar, G. Mirkowska, H. Rasiowa and A. Salwicki, *An Introduction to Algorithmic Logic, Mathematical Investigation in the Theory of Program*, Banach Center Publ. **2** (PWN, Warsaw, 1977).

[2] H. Friedman, Algorithmic procedures, generalized Turing algorithms and elementary recursion theory, in: *Logic Colloquium 69* (North-Holland, Amsterdam, 1971) 361-389.

[3] P.G. Hinman, *Recursion Theoretic Hierarchies* (Springer, New York, 1978).

[4] S. Mazur, Computable analysis, *Dissertationes Math.* **XXXIII** (1963).

[5] J.R. Shoenfield, *Mathematical Logic* (Addison-Wesley, Reading, MA, 1967).

[6] K. Skandalis, Programmable real numbers and functions, *Fund. Inform.* **VII** (1) (1984) 27-56.

[7] K. Skandalis, Programmability in the set of real numbers and second-order recursion, *Fund. Inform.* **VI** (3-4) (1983) 257-274.

[8] K. Weihrauch and C. Kreitz, Representations of the real numbers and the open subsets of the set of real numbers, *Ann. Pure Appl. Logic* **35** (1987) 247-260.